Induced Bases of Symmetry Classes of Tensors*

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ABSTRACT

Let $V^{m\otimes}$ denote the mth tensor power of the finite dimensional complex vector space V. Let $V_\chi(G)\subset V^{m\otimes}$ be the symmetry class of tensors corresponding to the permutation group G and the irreducible character χ of G. Each basis of V induces, in a natural way, a basis of $V^{m\otimes}$. The article considers the corresponding problem of inducing bases of $V_\chi(G)$.

INTRODUCTION

Let V be a complex inner product space of dimension n. Denote by $V^{m\otimes}$ the mth tensor power of V, and express a typical decomposable tensor as $v_1\otimes v_2\otimes\cdots\otimes v_m$. The inner product on V induces an inner product on $V^{m\otimes}$ which is determined by

$$(v_1\otimes v_2\otimes\cdots\otimes v_m, w_1\otimes w_2\otimes\cdots\otimes w_m)=\prod_{t=1}^{m}(v_t,w_t).$$

For each permutation σ in the symmetric group S_m, let $P(\sigma)$ be the (unique) linear operator on $V^{m\otimes}$ determined by the action $P(\sigma^{-1})v_1\otimes v_2\otimes\cdots\otimes v_m=v_{\sigma(1)}\otimes v_{\sigma(2)}\otimes\cdots\otimes v_{\sigma(m)}$. It easily follows that $P(\sigma^{-1})=P(\sigma)^{-1}=P(\sigma)^*$, the adjoint of $P(\sigma)$ with respect to the induced inner product. Suppose G is a subgroup of S_m, and let χ be an irreducible character of G. Define

$$T(G,\chi)=\frac{\chi(\mathrm{id})}{o(G)}\sum_{\sigma\in G}\chi(\sigma)P(\sigma).$$

*The research leading to this article was supported by the National Science Foundation through Grant No. MCS77-28437.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/81/050103+8$02.50

Then $T(G,\chi)$ is an orthogonal projection onto the symmetry class of tensors $V_\chi(G)$.

Let $E=\{e_1,e_2,\dots,e_n\}$ be an orthonormal basis of V. Then $\{e_\gamma^\otimes=e_{\gamma(1)}\otimes e_{\gamma(2)}\otimes\cdots\otimes e_{\gamma(m)}:\gamma\in\Gamma\}$ is an orthonormal basis of $V^{m\otimes}$, where Γ is the set of functions from $\{1,2,\dots,m\}$ into $\{1,2,\dots,n\}$ [3]. It follows that $\{e_\gamma^*=T(G,\chi)e_\gamma^\otimes:\gamma\in\Gamma\}$ must span $V_\chi(G)$. We are interested in obtaining a subset $\hat{\Delta}$ of Γ such that $\{e_\gamma^*:\gamma\in\hat{\Delta}\}$ is a basis of $V_\chi(G)$. Our first step will be the elimination of the zeros from $\{e_\gamma^*:\gamma\in\Gamma\}$.

Say that $\alpha,\beta\in\Gamma$ are equivalent, and write $\alpha\equiv\beta \pmod G$, if there is a $\sigma\in G$ such that $\beta=\alpha\sigma$. It is proved in [7] that

$$(e_\beta^*,e_\alpha^*)=\begin{cases}0 & \text{if } \beta\not\equiv\alpha \pmod G,\\ \dfrac{\chi(\mathrm{id})}{o(G)}\displaystyle\sum_{\pi\in G_\alpha}\chi(\pi\sigma) & \text{if } \beta=\alpha\sigma,\end{cases}\tag{1}$$

where $G_\alpha=\{\tau\in G:\alpha\tau=\alpha\}$. Taking $\beta=\alpha$ (and $\sigma=\mathrm{id}$) in (1), we obtain

$$(e_\alpha^*,e_\alpha^*)=\frac{\chi(\mathrm{id})}{o(G)}\sum_{\pi\in G_\alpha}\chi(\pi).\tag{2}$$

In particular, $e_\alpha^*\neq 0$ if and only if

$$\alpha\in\Omega=\left\{\gamma\in\Gamma:\sum_{\pi\in G_\gamma}\chi(\pi)\neq 0\right\},$$

i.e., $e_\alpha^*\neq0$ if and only if the restriction of χ to G_α contains the trivial character as a component. Our search for $\hat{\Delta}$ may therefore be restricted to Ω.

Now , if $\alpha\equiv\beta \pmod G$, then G_α is conjugate to G_β in G. It follows that Ω is a union of equivalence classes. Let $\bar{\Delta}$ be a system of distinct representatives for the equivalence classes in Ω. Then

$$\Omega=\bigcup_{\alpha\in\bar{\Delta}}\{\alpha\sigma:\sigma\in G\}.\tag{3}$$

THEOREM A [9]. *Let $E=\{e_1,e_2,\dots,e_n\}$ be a basis of V. Then $V_\chi(G)$ is the direct sum of the spaces $\langle e_{\alpha\sigma}^*:\sigma\in G\rangle$, as α ranges over $\bar{\Delta}$.*

Proof. As we may endow V with the (unique) inner product with respect to which E is orthonormal, the result is a restatement of (3). ■

In case $\chi(\mathrm{id})=1$, $e^*_{\alpha\sigma}=\chi(\sigma)e^*_\alpha$. Thus, each subspace in the direct sum of Theorem A is one dimensional. We have proved the following:

THEOREM B [5]. *Suppose $\chi(id)=1$. Let $E=\{e_1,e_2,\ldots,e_n\}$ be a basis of V. Then $\{e^*_\alpha : \alpha\in\bar{\Delta}\}$ is a basis of $V_\chi(G)$.*

In general, we have the following result of R. Freese.

THEOREM C [1]. *If $\gamma\in\Gamma$, then $\dim\langle e^*_{\gamma\sigma} : \sigma\in G\rangle=\chi(\mathrm{id})(\chi,1)_{G_\gamma}$, i.e., the degree of χ times the number of occurrences of the trivial character in the restriction of χ to G_γ.*

In principle, for each $\alpha\in\bar{\Delta}$, we may select a subset L_α of $\{\alpha\sigma : \sigma\in G\}$ such that $\{e^*_\gamma : \gamma\in L_\alpha\}$ is a basis of $\langle e^*_{\alpha\sigma} : \sigma\in G\rangle$. By putting these subsets together, we obtain a set

$$\hat{\Delta}=\bigcup_{\alpha\in\bar{\Delta}} L_\alpha$$

such that $\{e^*_\gamma : \gamma\in\hat{\Delta}\}$ is a basis of $V_\chi(G)$. It can be shown that $\hat{\Delta}$ is independent of E.

RESULTS

As a preliminary step in the analysis of the linear relations among the elements of $\{e^*_{\alpha\sigma} : \sigma\in G\}$ it seems natural to consider

$$G^\alpha=\{\sigma\in G : \text{there is a number, } c_\alpha(\sigma)\text{, such that } e^*_{\alpha\sigma}=c_\alpha(\sigma)e^*_\alpha\}.$$

Of course, $G_\alpha\subseteq G^\alpha$, and the restriction of c_α to G_α is identically equal to 1. It was shown in [7] that G^α does not depend on E, that G^α is a subgroup of G, and that c_α is a character of G^α of degree 1. Moreover, if $\chi(\mathrm{id})=1$, then $G^\alpha=G$ and $c_\alpha=\chi$.

THEOREM 1. *Let $\sigma\in G$ and $\alpha\in\Omega$. Then $\sigma\in G^\alpha$ if and only if*

$$(\chi,1)_{G_\alpha}=\left|\frac{1}{o(G_\alpha)}\sum_{\pi\in G_\alpha}\chi(\pi\sigma)\right|. \tag{4}$$

Proof. Suppose that Eq. (4) holds. According to (1),

$$(e^*_{\alpha\sigma}, e^*_\alpha) = \frac{\chi(\mathrm{id})}{o(G)} \sum_{\pi \in G_\alpha} \chi(\pi\sigma).$$

Thus $|(e^*_{\alpha\sigma}, e^*_\alpha)| = \chi(\mathrm{id})(\chi, 1)_{G_\alpha}/[G: G_\alpha]$. But, this is exactly the value given by (2) for (e^*_α, e^*_α). On the other hand, since $P(\tau)$ is unitary, $\tau \in S_m$, it follows that $(e^*_{\alpha\sigma}, e^*_{\alpha\sigma}) = (e^*_\alpha, e^*_\alpha)$. Putting this information together yields $|(e^*_{\alpha\sigma}, e^*_\alpha)| = \| e^*_{\alpha\sigma} \| \cdot \| e^*_\alpha \|$, i.e., the case of equality in the Cauchy-Schwarz Inequality. Since $e^*_\alpha \neq 0$, it follows that $e^*_{\alpha\sigma}$ is multiple of e^*_α and, hence, that $\sigma \in G^\alpha$.

Conversely,

$$\begin{aligned} |(e^*_{\alpha\sigma}, e^*_\alpha)| &= |(c_\alpha(\sigma) e^*_\alpha, e^*_\alpha)| \\ &= |c_\alpha(\sigma)| \left| \frac{\chi(\mathrm{id})}{o(G)} \sum_{\pi \in G_\alpha} \chi(\pi) \right| \\ &= \frac{[G: G_\alpha]}{\chi(\mathrm{id})(\chi, 1)_{G_\alpha}}, \end{aligned}$$

since $|c_\alpha(\sigma)| = 1$. Comparing with the value of $|(e^*_{\alpha\sigma}, e^*_\alpha)|$ given by (1), we obtain the result. ∎

THEOREM 2. *Suppose $\alpha \in \Omega$. If $\sigma \in G^\alpha$, then $(\chi, 1)_{G_\alpha} = (\chi, c_\alpha)_H$, where $H = \langle G_\alpha, \sigma \rangle$, the group generated by the elements of G_α together with σ.*

Proof. Recall that the restriction of c_α to G_α is identically 1. Since G_α is a subgroup of H, it follows that

$$(\chi, 1)_{G_\alpha} = (\chi, c_\alpha)_{G_\alpha} \geqslant (\chi, c_\alpha)_H.$$

It remains to establish the reverse inequality. Let $\{A(\tau) = (a_{ij}(\tau)) : \tau \in G\}$ be an irreducible, unitary representation of G which affords χ. Without loss of generality, we may assume that the restriction of this representation to G_α is fully reduced and that the trivial representation occurs first with multiplicity $r = (\chi, 1)_{G_\alpha}$, i.e.,

$$A(\pi) = \begin{pmatrix} I_r & 0 \\ 0 & B(\pi) \end{pmatrix}, \tag{5}$$

$\pi \in G_\alpha$. As in the proof of Theorem 1, we may combine $e^*_{\alpha\sigma} = c_\alpha(\sigma)e^*_\alpha$ and the information provided by Eq. (1) to obtain

$$\sum_{\pi\in G_\alpha} \chi(\pi\sigma) = c_\alpha(\sigma) \sum_{\pi\in G_\alpha} \chi(\pi) = ro(G_\alpha)c_\alpha(\sigma). \tag{6}$$

But

$$\sum_{\pi\in G_\alpha} \chi(\pi\sigma) = \sum_{i,j=1}^{\chi(\mathrm{id})} \left[\sum_{\pi\in G_\alpha} a_{ij}(\pi) \right] a_{ji}(\sigma).$$

By the Schur relations [2, p. 32; 10, p.16], the term in brackets is zero if $i \neq j$ or if $i > r$. For $1 \leqslant i = j \leqslant r$, it is $o(G_\alpha)$. Thus

$$\sum_{\pi\in G_\alpha} \chi(\pi\sigma) = o(G_\alpha) \sum_{i=1}^{r} a_{ii}(\sigma). \tag{7}$$

Comparing (6) and (7) yields

$$rc_\alpha(\sigma) = \sum_{i=1}^{r} a_{ii}(\sigma).$$

Since $\{A(\tau): \tau \in G\}$ is a unitary representation, $|a_{ii}(\sigma)| \leqslant 1$ for all i. Since $|c_\alpha(\sigma)| = 1$, it must be that $a_{ii}(\sigma) = c_\alpha(\sigma)$, $1 \leqslant i \leqslant r$. Therefore

$$A(\sigma) = \begin{pmatrix} c_\alpha(\sigma)I_r & 0 \\ 0 & B(\sigma) \end{pmatrix}.$$

It follows that

$$A(\tau) = \begin{pmatrix} c_\alpha(\tau)I_r & 0 \\ 0 & B(\tau) \end{pmatrix}$$

for all $\tau \in H = \langle G_\alpha, \sigma \rangle$. Hence, $(\chi, c_\alpha)_H \geqslant r$.

COROLLARY 1. *Let $\sigma \in G$ and $\alpha \in \Omega$. Then $e^*_{\alpha\sigma} = e^*_\alpha$ if and only if $(\chi, 1)_{G_\alpha} = (\chi, 1)_H$, where $H = \langle G_\alpha, \sigma \rangle$.*

Proof. If $e^*_{\alpha\sigma}=e^*_\alpha$, then $\sigma\in G^\alpha$ and $c_\alpha(\sigma)=1$. Since c_α is a homomorphism, it follows that $c_\alpha(\tau)=1$ for all $\tau\in H$, and necessity follows from Theorem 2. To prove sufficiency, suppose $(\chi,1)_{G_\alpha}=(\chi,1)_H=r$. Let $\{A(\tau):\tau\in G\}$ be an irreducible, unitary representation of G which affords χ. Without loss of generality we may assume $A(\pi)$ takes the form (5) for all $\pi\in H$. Suppose $\{B(\pi):\pi\in H\}$ affords the character λ. Then, in particular,

$$(\lambda,1)_H=(\lambda,1)_{G_\alpha}=0. \tag{8}$$

It follows from (1) that

$$(e^*_{\alpha\sigma},e^*_\alpha)=\frac{\chi(\mathrm{id})}{o(G)}\sum_{\pi\in G_\alpha}\chi(\pi\sigma) \tag{9}$$

$$=\frac{\chi(\mathrm{id})}{[G:G_\alpha]}\left(r+\frac{1}{o(G_\alpha)}\sum_{\pi\in G_\alpha}\lambda(\pi\sigma)\right). \tag{10}$$

We next show that the right hand side of (10) is $r\chi(\mathrm{id})/[G:G_\alpha]$. Let

$$S=\sum_{\pi\in G_\alpha}B(\pi).$$

Then $S=S^*$ and $o(G_\alpha)S=S^2$. It follows that S is positive semidefinite hermitian. But, by (8), trace $S=0$. Therefore $S=0$. Finally,

$$\sum_{\pi\in G_\alpha}\lambda(\pi\sigma)=\mathrm{trace}(SB(\sigma))$$
$$=0.$$

Returning to (10), we find

$$(e^*_{\alpha\sigma},e^*_\alpha)=\frac{[G:G_\alpha]}{\chi(\mathrm{id})(\chi,1)_{G_\alpha}}. \tag{11}$$

It follows, as in the proof of Theorem 1, that there is a number c such that $e^*_{\alpha\sigma}=ce^*_\alpha$. Then $(e^*_{\alpha\sigma},e^*_\alpha)=c(e^*_\alpha,e^*_\alpha)$. On the other hand, we have just seen that $(e^*_{\alpha\sigma},e^*_\alpha)=(e^*_\alpha,e^*_\alpha)$. Therefore, $c=1$. ∎

If α is one to one, then $G_\alpha=\{\mathrm{id}\}$ and $\langle G_\alpha,\sigma\rangle=\langle\sigma\rangle$. Thus, Corollary 1 may be thought of as an extension of the following: "If α is one to one, then

$e^*_{\alpha\sigma}=e^*_\alpha$ if and only if $\chi(\sigma)=\chi(\mathrm{id})$." An extension of this statement in another direction may be found in [11]. (Also see [4].)

COROLLARY 2. *Suppose $\chi(\tau)=(\chi,1)_{G_\alpha}$ for all τ in the coset $G_\alpha\sigma$. Then $e^*_{\alpha\sigma}=e^*_\alpha$.*

Proof. Return to Eq. (9) above. The hypothesis immediately yields (11), and we may proceed from that point exactly as in the proof of Corollary 1. ∎

EXAMPLE. Let $G=S_5$. Suppose χ is the irreducible character of S_5 corresponding to the partition $5=3+2$. Think of Γ as a set of integer sequences of length m, i.e., $\gamma\in\Gamma$ corresponds to the sequence $(\gamma(1),\gamma(2),\dots,\gamma(m))$. Let $\alpha=(1,1,1,2,3)$, and let σ be the transposition (45). Then $\alpha\sigma=(1,1,1,3,2)$. It was discovered by a brute force computation, and reported in [7], that $e^*_{\alpha\sigma}=e^*_\alpha$. It turns out that $(\chi,1)_{G_\alpha}=1$ and $\chi(\pi\sigma)=1$ for all $\pi\in G_\alpha=S_3$.

We conclude with a result which was stated but not proved in [7].

THEOREM 3. *Let G be a subgroup of S_m. Suppose χ is an irreducible character of G. If $\alpha\in\Omega$, then $(\chi,c_\alpha)_{G^\alpha}\neq 0$, i.e., $(\chi,1)_{G_\alpha}\neq 0$ always implies $(\chi,c_\alpha)_{G^\alpha}\neq 0$.*

Proof. Let $\{A(\tau)=(a_{ij}(\tau))\colon \tau\in G\}$ be a representation which affords χ. Assume that the restriction of this representation to G^α is fully reduced. Let $E=\{e_1,e_2,\dots,e_n\}$ be a basis of V. Now, not only is $T(G,\chi)$ idempotent, but it commutes with $P(\tau)$, $\tau\in G$. Therefore, since $\alpha\in\Omega$,

$$\begin{aligned}0\neq\frac{o(G)}{\chi(\mathrm{id})}e^*_\alpha&=\sum_{\tau\in G}\chi(\tau^{-1})e^*_{\alpha\tau}\\&=\sum_{j=1}^{k}\sum_{\tau\in G^\alpha}\chi\left(\tau_j^{-1}\tau^{-1}\right)e^*_{\alpha\tau\tau_j},\end{aligned}$$

where $\tau_1,\tau_2,\dots,\tau_k$ are representatives of the distinct right cosets of G^α in G. Thus,

$$\begin{aligned}0&\neq\sum_{j=1}^{k}\sum_{\tau\in G^\alpha}\chi\left(\tau_j^{-1}\tau^{-1}\right)c_\alpha(\tau)e^*_{\alpha\tau_j}\\&=\sum_{j=1}^{k}\sum_{s,t=1}^{\chi(\mathrm{id})}a_{st}\left(\tau_j^{-1}\right)\left[\sum_{\tau\in G^\alpha}a_{ts}(\tau^{-1})c_\alpha(\tau)\right]e^*_{\alpha\tau_j}.\end{aligned}$$

Now, if c_α is not a component of the restriction of χ to G^α, then by the Schur relations, the term in brackets is zero for all s and t, i.e., a contradiction. ∎

Since $G^\alpha = G$ and $\chi = c_\alpha$ when χ is of degree 1, the converse to Theorem 3 fails.

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Received 12 November 1979